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Solvability of a superlinear Neumann problem at resonance in the first eigenvalue*

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Abstract: The existence of solutions for a class of superlinear second-order discrete Neumann boundary value problems of the form

$$\begin{cases} -\Delta^2 u(t-1) = \left(u^+(t)\right)^p + f(t), & t \in [1, T]_{\mathbb{Z}}, \\ \Delta u(0) = \Delta u(T) = 0, \end{cases} \tag{P}$$

where $T \geqslant 2$ is an integer, $p > 1$, and f satisfies the condition $\sum_{t=1}^{T} f(t) < 0$, is investigated. Based on the topological degree theory, we show that the problem (P) possesses at least one solution. Additionally, we extend our analysis to systems of second-order discrete equations with similar nonlinearities.

Key words: discrete equations; resonance; superlinearity; topological degree

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Let $T \geqslant 2$ be an integer, $\mathbb{T} := \{1,2,\cdots,T\}, \hat{\mathbb{T}} := \{0,1,\cdots,T+1\}$. In this paper, we are concerned with the solvability of the Neumann problem

$$\begin{cases} -\Delta^2 u(t-1) = \left(u^+(t)\right)^p + f(t), & t \in \mathbb{T}, \\ \Delta u(0) = \Delta u(T) = 0, \end{cases} \tag{1}$$

where $p > 1$ is a constant, $\Delta u(t) = u(t+1) - u(t)$ is the forward difference operator, and $\Delta^2 u(t) = \Delta\left(\Delta u(t)\right)$. The term $u^+(t) = \max\{u(t), 0\}$ represents the positive part of $u(t)$, and $f(t)$ is a real-valued function satisfying the condition

(H1) $f: \mathbb{T} \to \mathbb{R}$ and satisfies $\sum_{t=1}^{T} f(t) < 0$.

Since $\lambda_1 = 0$ is the first eigenvalue of the $-\Delta^2 u(t-1)$ with Neumann boundary condition, the problem (1) is said to be resonance on λ_1 at $u \to -\infty$. Specifically, the resonance at $\lambda_1 = 0$ as $u \to -\infty$ occurs because the term $\left(u^+\right)^p$ vanishes for large negative u, causing the ratio of the nonlinearity to the variable u to asymptotically approach the first eigenvalue $\lambda_1 = 0$.

Discrete boundary value problems like (1) arise naturally in various fields such as population dynamics, neural networks, and social economic systems, where the state variables are often sampled at discrete time intervals. Physically, the operator $-\Delta^2 u(t-1)$ can be viewed as the discrete counterpart of the elastic beam equation or the heat conduction model in a one-dimensional discrete medium. The Neumann boundary conditions $\Delta u(0) = \Delta u(T) = 0$ physically represent "no-flux" or "insulated" ends, which are common in thermal

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equilibrium problems.

The existence of solutions for second-order differential equations with Neumann boundary conditions has been a widely explored topic in recent years. In particular, the following continuous Neumann problem

$$\begin{cases} -\mathcal{L}u(x) = \left(u^+(x)\right)^p + h(x), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $p > 1, \Omega \subset \mathbb{R}^N (N \geqslant 1)$ is a smooth, bounded domain, and $\mathcal{L}u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian operator, has been extensively studied in the literature. Notice that, the nonlinearity $\left(u^+(x)\right)^p$ exhibits resonance at $-\infty$ and superlinear growth at $+\infty$. For a more detailed discussion of the theory, we refer to the works (Ward,1982; Arcoya et al. ,1995; Papageorgiou et al. ,2013; de Paiva et al. ,2017; Liu et al. ,2024).

However, while the continuous problem has been well-studied, the corresponding discrete analog (1) has received less attention. Problem (1) is a discretization of the continuous problem (2) in the case $N = 1$. As such, solving (1) can yield numerical insights that are valuable for understanding solutions to the continuous problem (2). Thus, it is both of mathematical significance and of practical importance to study the solvability of problem (1). For other related results on nonlinear discrete equations with various boundary conditions, we refer the reader to (Zhu et al. ,2009; Liu et al. ,2011; Wang et al. ,2013; Ma et al. ,2019; Freedman et al. ,2022; Gao et al. , 2024).

Notice that, Bai et al. (2015) applied Guo-Krasnosel'skii fixed point theorem to study the existence of positive solutions for the discrete Neumann boundary value problem

$$\begin{cases} -\Delta^2 u(t-1) = g\left(t, u(t)\right), \quad t \in \mathbb{T}, \\ \Delta u(0) = \Delta u(T) = 0, \end{cases}$$

where $g: \mathbb{T} \times \mathbb{R}^+ \to \mathbb{R}$ is a continuous function and $g\left(t, u\right)$ is assumed to have at least linear growth. That is there exists a constant number $L > 0$ such that

$$g\left(t, u(t)\right) + Lu(t) + m(t) \geqslant 0, \quad (t, u) \in \mathbb{T} \times \mathbb{R}^+.$$

On this basis, Long et al. (2018) further investigated the existence of solutions when the nonlinear term $g\left(t, u\right)$ has at most linear growth using variational methods.

Compared to these existing works, problem (1) presents several new and unresolved challenges:

(i) Non-invertibility: The resonance at $\lambda_1 = 0$ implies that the linear operator is singular. The solution must be sought in a setting where the kernel space (constant functions) interacts strongly with the nonlinearity.

(ii) Asymmetric Growth: The nonlinearity $\left(u^+\right)^p$ exhibits superlinear growth at $+\infty$ (since $p > 1$) but vanishes at $-\infty$. This asymmetry, combined with resonance, makes it difficult to apply standard fixed-point theorems or variational techniques directly.

(iii) Discrete Constraints: Unlike continuous differential equations, discrete difference equations lack certain classical tools such as Rolle's theorem and standard Sobolev embeddings, which are crucial for establishing a priori bounds in the continuous case.

To overcome these difficulties, we adapt the approach of Cuesta et al. (2003) to the discrete setting. By combining a priori estimates with topological degree theory, we handle the resonance at λ_1 using a one-sided Landesman-Lazer condition (H1). The main results are as follows:

The main results of this paper are as follows:

Theorem 1 Let $p > 1$ and f satisfies the hypothesis (H1). Then, problem (1) possesses at least one solution.

Since we will use topological arguments to prove Theorem 1, we shall need a priori bound on the solutions of problem (1).

Theorem 2　Assume that hypothesis (H1) holds, and let $u:\hat{\mathbb{T}}\to\mathbb{R}$ be a solution of problem (1). Then, there exists a constant C, depending only on p and T, such that

$$\|u\|_{\infty}<C,$$

where $\|u\|_{\infty}=\max_{t\in\hat{\mathbb{T}}}|u(t)|$.

We also study the solvability of the system

$$\begin{cases}-\Delta^2 u(t-1)=\left(v^{+}(t)\right)^{p}+f(t), & t\in\mathbb{T},\\ -\Delta^2 v(t-1)=\left(u^{+}(t)\right)^{q}+g(t), & t\in\mathbb{T},\\ \Delta u(0)=\Delta u(T)=0,\\ \Delta v(0)=\Delta v(T)=0,\end{cases}\tag{3}$$

where f, g satisfy hypothesis (H1), and the exponents $p, q>1$. We get following results.

Theorem 3　Let $p, q>1$ and f, g satisfy the hypothesis (H1). Then problem (3) possesses at least one solution.

1　A priori estimates

Let

$$X=\left\{u\,\middle|\,u:\hat{\mathbb{T}}\to\mathbb{R},\Delta u(0)=\Delta u(T)=0\right\}$$

be a function space endowed with the norm

$$\|u\|_{\infty}=\max_{t\in\hat{\mathbb{T}}}|u(t)|.$$

Since $\lambda_1=0$ and the corresponding eigenfunction is $\varphi_1=1$, for $u, v\in X$, we can decompose u and v as follows:

$$u=\bar{u}+\tilde{u},\quad v=\bar{v}+\tilde{v},\tag{4}$$

where $\bar{u},\bar{v}\in\mathbb{R}$ are constants, and $\sum_{s=1}^{T}\tilde{u}(s)=\sum_{s=1}^{T}\tilde{v}(s)=0$.

Proof of Theorem 2　Multiplying the equation (1) by $\varphi_1=1$, we perform a direct computation to obtain

$$\sum_{t=1}^{T}\left(u^{+}(t)\right)^{p}=-\sum_{t=1}^{T}f(t)\leqslant(T-1)\|f\|_{\infty}.$$

Using Hölder inequality with p and $q=\frac{p}{p-1}$, we get

$$\sum_{t=1}^{T}u^{+}(t)\cdot 1\leqslant\left(\sum_{t=1}^{T}\left(u^{+}(t)\right)^{p}\right)^{\frac{1}{p}}\left(\sum_{t=1}^{T}1^{q}\right)^{\frac{1}{q}}\leqslant C,$$

which implies that

$$\|u^{+}\|_{\infty}<C_0.$$

From equation (1) and the decompositio (4), we have

$$-\Delta^2\tilde{u}(t-1)=\left(u^{+}(t)\right)^{p}+f(t),$$

so

$$\left|\Delta^2\tilde{u}(t-1)\right|\leqslant\left|\left(u^{+}(t)\right)^{p}\right|+|f(t)|\leqslant C.$$

Together with the boundary condition $\Delta\tilde{u}(0)=0$, this implies

$$\|\Delta u\|_{\infty}=\|\Delta\tilde{u}\|_{\infty}<C_1.$$

By (1), we obtain

$$0=\Delta u(0)-\Delta u(T)=\sum_{t=1}^{T}-\Delta^2 u(t-1)=\sum_{t=1}^{T}\left[\left(u^{+}(t)\right)^{p}+f(t)\right].\tag{5}$$

Conditions (H1) and equation (5) imply that for $t \in \mathbb{T}$,

$$u^{+}(t) \not\equiv 0.$$

Therefore, there exists t_0 such that $u(t_0) > 0$, and we deduce

$$0 < u(t_0) < C_0.$$

Let $\xi \in [t_0, t]_{\mathbb{Z}}$ such that $|\Delta u(\xi)| = \max_{k \in [t_0, t]_{\mathbb{Z}}} |\Delta u(k)|$. Now applying the difference mean value theorem, we get

$$|u(t) - u(t_0)| = \left|\sum_{k=t_0}^{t-1} \Delta u(k)\right| \leqslant \sum_{k=t_0}^{t-1} |\Delta u(k)| \leqslant \sum_{k=t_0}^{t-1} |\Delta u(\xi)| = |\Delta u(\xi)|(t - t_0).$$

Subsequently, we have

$$|u(t)| \leqslant u(t_0) + |\Delta u(\xi)|(t - t_0) \leqslant C_1 T + C_0.$$

Therefore

$$\| u \|_{\infty} < C. \qquad \square$$

2 Proof of Theorem 1

Define the operator $L: X \to X$ as

$$(Lu)(t) := -\Delta^2 u(t-1) + u(t).$$

Let $T_f: X \to X$ be the map defined by

$$T_f(u) = L^{-1}\left((u^{+}(t))^{p} + u(t) + f(t)\right).$$

Since T_f is a compact, continuous map, it follows that $(T_f u)(t) = u \Leftrightarrow u$ if and only if u solves equation(1), see (Mawhin, 1973; Agarwal, 2000). We will now use the Brouwer degree, denoted $d(\cdot,\cdot,\cdot)$, in the following.

Proposition 1 Assume that (H1) holds. For any $\epsilon > 0$, there exists $\delta > 0$, such that for any solution u of (1) with $\| f \|_{\infty} < \delta$, we have

$$\| u^{+} \|_{\infty} < \epsilon.$$

Proof Assume, for contradiction, that there exist $\epsilon_0 > 0$ and f_n with $\| f_n \|_{\infty} < 1/n$, such that for each n, the solution u_n of (1) corresponding to $f = f_n$ satisfies

$$\| u_n^{+} \|_{\infty} \geqslant \epsilon_0. \tag{6}$$

Then, the following system holds for each n,

$$\begin{cases} -\Delta^2 u_n(t-1) = (u_n^{+}(t))^{p} + f_n(t), & t \in \mathbb{T}, \\ \Delta u_n(0) = \Delta u_n(T) = 0. \end{cases}$$

Moreover,

$$\sum_{t=1}^{T} (u_n^{+}(t))^{p} = -\sum_{t=1}^{T} f_n(t) \to 0 \quad \text{as } n \to \infty. \tag{7}$$

On the other hand, Theorem 2 gives

$$\| u_n^{+} \|_{\infty} \leqslant C.$$

After taking a subsequence if necessary, we have,

$$u_n^{+} \to u_*^{+}.$$

Thus, from (6), it follows that

$$\| u_*^{+} \|_{\infty} \geqslant \epsilon_0. \tag{8}$$

Combining (7) and (8), and using (H1), we deduces that

$$\sum_{t=1}^{T} (u_*^{+}(t))^{p} > 0.$$

However, this contradicts (7) for sufficiently large n. $\square$

Proposition 2 There exist $\varepsilon > 0$ and $R_0 > 0$ such that for all functions f satisfying (H1) with $\|f\|_\infty < \varepsilon$, for which problem (1) possesses at least one solution, it follows that

$$d\left(\mathrm{Id} - T_f, (0, R), 0\right) \neq 0$$

for all $R \geqslant R_0$.

Proof We first prove that there exists ε such that for all f with $\|f\|_\infty < \varepsilon$, any solution u_0 of problem (1) is non-degenerate with a Morse index equal to 1. Let ϵ be the function provided by Proposition 1, and let $\varepsilon < 1$ such that

$$\epsilon < \left(\frac{\lambda_2}{p}\right)^{1/(p-1)}.$$

Let f be a function satisfying (H1), with $\|f\|_\infty \leqslant \varepsilon$. By Proposition 1, any solution u_0 of (1) satisfies

$$\|u_0^+\|_\infty \leqslant \epsilon < \left(\frac{\lambda_2}{p}\right)^{1/(p-1)}. \tag{9}$$

Thus, there exists R_0 such that $\|u_0\|_\infty < R_0$.

The linearized problem of (1) at u_0 is given by

$$\begin{cases} -\Delta^2 v(t-1) = p\left(u_0^+(t)\right)^{p-1} v, & t \in \mathbb{T}, \\ \Delta v(0) = \Delta v(T) = 0. \end{cases} \tag{10}$$

Denote the eigenvalues of the eigenvalue problem

$$\begin{cases} -\Delta^2 v(t-1) = \mu a(t) v(t), & t \in \mathbb{T}, \\ \Delta v(0) = \Delta v(T) = 0 \end{cases}$$

by

$$0 = \mu_1(a) < \mu_2(a) < \cdots < \mu_T(a).$$

By (9), we have $0 = \lambda_1 < a(t) := p\left(u_0^+(t)\right)^{p-1} < \lambda_2$, a. e. , so

$$0 = \mu_1(a) < 1 < \mu_2(a).$$

Hence, $v \equiv 0$ is the unique solution of (10), and therefore u_0 is a non-degenerate solution of (1) with Morse index equal to 1.

By estimate (9), the degree above is well defined for all $R \geqslant R_0$. Since all possible solutions of $u = T_f(u)$ are non-degenerated, they are isolated, and there is only a finite number of them in $(0, R)$. We recall that the index of each solution is equal to $(-1)^\beta$, where β is the Morse index. Therefore, we have

$$d\left(\mathrm{Id} - T_f, (0, R), 0\right) = \sum_{i=1}^{n} (-1)^{\beta_i} = \sum_{i=1}^{n} (-1) \neq 0. \qquad \square$$

Proof of Theorem 1 Choose $f_1(t) := -\left(s\varphi_1(t)\right)^p$ with $0 < s < \left(\frac{\varepsilon}{\|\varphi_1^p\|_\infty}\right)^{1/p}$, where ε is given by Proposition 2. Notice that $u(t) = s\varphi_1(t)$ is a solution of problem (1) for $f_1(t) = -\left(s\varphi_1(t)\right)^p$, and Proposition 1 applies. Thus

$$d\left(\mathrm{Id} - T_{f_1}, (0, R), 0\right) \neq 0$$

for sufficiently large R. Consider the following homotopy

$$\begin{cases} -\Delta^2 u(t-1) = \left(u^+(t)\right)^p + (1-\tau) f(t) + \tau f_1(t), & t \in \mathbb{T}, \\ \Delta u(0) = \Delta u(T) = 0, \end{cases} \tag{11}$$

where $0 \leqslant \tau \leqslant 1$. From the a-priori estimates of Theorem 2, all the solution of problem (11) are uniformly bounded in X by some constant depending on p and T. Hence, for any $R > \max\{R_0, C\}$, we have

$$d\left(\mathrm{Id} - T_f, (0, R), 0\right) = d\left(\mathrm{Id} - T_{f_1}, (0, R), 0\right) \neq 0,$$

and the conclusion of the theorem follows.

3 Proof of Theorem 3

We proceed as in the proof of Theorem 1. We begin by proving that the solutions (u,v) of problem (3) are non-degenerated with index 1, provided f and g are sufficiently small and satisfy the condition (H1). The linearized problem of (3) at some solution (u_0,v_0) is as follows

$$\begin{cases} -\Delta^2 u(t-1) = p\left(v_0^+(t)\right)^{p-1} v, & t\in\mathbb{T},\\ -\Delta^2 v(t-1) = q\left(u_0^+(t)\right)^{q-1} u, & t\in\mathbb{T},\\ \Delta u(0) = \Delta u(T) = 0,\\ \Delta v(0) = \Delta v(T) = 0. \end{cases}$$

The non-degeneracy of the solutions and the computation of the index will follow directly from the results of the two lemmas below.

Lemma 1 There exists $\varepsilon > 0$ such that for any $a, b:\mathbb{T}\to\mathbb{R}$, with $a(t), b(t) \geqslant 0$ almost everywhere, a, b not identically zero, and for some constant $c\in\mathbb{R}$ and $s\in[0,1]$. If $\|a\|_\infty < \varepsilon$, $\|b\|_\infty < \varepsilon$, and $0 < c < \varepsilon$, the system

$$\begin{cases} -\Delta^2 u(t-1) = sa(t)v(t) + (1-s)cv(t), & t\in\mathbb{T},\\ -\Delta^2 v(t-1) = sb(t)u(t) + (1-s)cu(t), & t\in\mathbb{T},\\ \Delta u(0) = \Delta u(T) = 0,\\ \Delta v(0) = \Delta v(T) = 0, \end{cases} \tag{12}$$

only the trivial solution $u = v = 0$.

Proof Suppose, by contradiction, that for all $\varepsilon_n = 1/n$, there exist sequences of functions $a_n(t), b_n(t)$ not identically zero, satisfying

$$0 \leqslant a_n(t) \leqslant \frac{1}{n}, \qquad 0 \leqslant b_n(t) \leqslant \frac{1}{n},$$

and sequences of real numbers $s_n\in[0,1]$, $c_n\in(0, 1/n]$ such that the system (12) has non-trivial solution (u_n, v_n) for these choices of coefficients.

Consider the sequences

$$\tilde{u}_n = \frac{u_n}{\|u_n\|_\infty} \text{ and } \tilde{v}_n = \frac{v_n}{\|v_n\|_\infty}.$$

We now explore two possible cases for the ratior $\dfrac{\|v_n\|_\infty}{\|u_n\|_\infty}$.

(i) If $\dfrac{\|v_n\|_\infty}{\|u_n\|_\infty}$ is bounded.

In this case, we consider

$$\begin{cases} -\Delta^2 \tilde{u}_n(t-1) = \left(s_n a_n(t) + (1-s_n)c_n\right)\dfrac{v_n(t)}{\|u_n\|_\infty}, & t\in\mathbb{T},\\ \Delta\tilde{u}_n(0) = \Delta\tilde{u}_n(T) = 0. \end{cases}$$

Since $\|\tilde{u}_n\|_\infty = 1$, we conclude that there exists $\tilde{u}\in X$ such that, after passing to a subsequence if necessary, we can assume

$$\tilde{u}_n \to \tilde{u}.$$

It follows that $\|\tilde{u}\|_\infty = 1$. Taking the limit in the equation for $\tilde{u}_n$, we obtain

$$\begin{cases} -\Delta^2\tilde{u}(t-1) = 0, & t\in\mathbb{T},\\ \Delta\tilde{u}(0) = \Delta\tilde{u}(T) = 0. \end{cases}$$

Therefore, $\tilde{u} = \pm\varphi_1$, implying $\tilde{u}_n \to \pm\varphi_1$ as n grows large.

Next, testing the second equation of system (12) with $\varphi_1 = c$, we deduce that

$$\sum_{t=1}^{T}\left(s_n b_n(t) + (1 - s_n)c_n\right)u_n(t) = 0,$$

This leads to a contradiction, as $\tilde{u}_n$ has a defined signal, and the term multiplying it is non-negative and non-trivial.

(ii) If $\dfrac{\|v_n\|_\infty}{\|u_n\|_\infty}$ is unbounded.

In this case, the sequence $\dfrac{\|u_n\|_\infty}{\|v_n\|_\infty}$ is bounded, and we follow the same argument as in case (i), but with the roles of u_n and v_n reversed.

Thus, we conclude that the only solution to system (12) is the trivial solution $u = v = 0$. □

Lemma 2 Let $\lambda_1 = 0 < c < \lambda_2$. Then eingevalue problem with parameter μ given by

$$\begin{cases} -\Delta^2 u(t-1) = \mu c v(t), & t \in \mathbb{T}, \\ -\Delta^2 v(t-1) = \mu c u(t), & t \in \mathbb{T}, \\ \Delta u(0) = \Delta u(T) = 0, \\ \Delta v(0) = \Delta v(T) = 0, \end{cases} \tag{13}$$

has a single eingevalue μ in the range $[0,1]$, given by

$$\mu_1 = 0.$$

Proof We express

$$u = \sum_{i=1}^{T} t_i \varphi_i, \qquad v = \sum_{i=1}^{T} s_i \varphi_i.$$

Testing the first equation of the system (13) with φ_j, we obtain

$$-\Delta^2\left(\sum_{i=1}^{T} t_i \varphi_i(t-1)\right)\varphi_j = \mu c\left(\sum_{i=1}^{T} s_i \varphi_i(t)\right)\varphi_j.$$

which simplifies to

$$t_j\lambda_j - \mu c s_j = 0.$$

Similarly, from the second equation, we get

$$\mu c t_j - \lambda_j s_j = 0.$$

To solve for the eigenfunctions of the eigenvalue problem (13), we must have the determinant

$$\begin{vmatrix} \lambda_j & -\mu c \\ \mu c & -\lambda_j \end{vmatrix} = 0,$$

which gives the eigenvalues

$$\mu_j = \pm\frac{\lambda_j}{c}.$$

Since $\lambda_1 = 0$ and $0 < c < \lambda_2$, we conclude that

$$0 = \mu_1 < 1 < \mu_2 = \frac{\lambda_2}{c}.$$

Thus, the result is proved. □

To prove Theorem 3, we follow similar topological arguments as those used in the proof of Theorem 1. In this case, a priori bound for the solutions of (3) are provided by the results below.

Lemma 3 Assume that (H1) holds, and let $(u, v) \in \mathbb{R} \times \mathbb{R}$ be a solution of the system (3). Then there exists a constant M, depending only on p, q and T, such that

$$\|u\|_\infty + \|v\|_\infty < M.$$

The proof of Lemma 3 is similar to the proof of Theorem 2.

Proof of Theorem 3 We begin by introducing the following formulation via fixed-point theory. Let $T_{(f,g)}$: $X \times X \to X \times X$ be the map:

$$T_{(f,g)}(u,v) = \left(L^{-1}\left(\left(v^+(t)\right)^p + u(t) + f(t)\right), L^{-1}\left(\left(u^+(t)\right)^q + v(t) + g(t)\right)\right).$$

It follows that $T_{(f,g)}$ is continuous, compact, and $T_{(f,g)}(u,v) = (u,v)$ if and only if (u,v) is a solution of (3).

Define f_1, g_1 as $f_1 = -(\eta\varphi_1)^p$ and $g_1 = -(\xi\varphi_1)^q$, with $\varphi_1 = c$ and $\eta, \xi > 0$. It is easy to verify that $(u,v) = (\xi\varphi_1, \eta\varphi_1)$ is a solution of (3) with (f_1, g_1). Furthermore, choosing ξ and η sufficiently small, we can assume that

$$\left\| p\left(v^+\right)^{p-1} \right\|_\infty < \varepsilon, \qquad \left\| q\left(u^+\right)^{q-1} \right\|_\infty < \varepsilon,$$

where ε is given by Lemma 1. Thus (u,v) is non-degenerate and, for some sufficiently small $R_{(u,v)}$,

$$d\left(\mathrm{Id} - T_{(f_1,g_1)}, R_{(u,v)} \times R_{(u,v)}, 0\right) = (-1)^{\beta(u,v)},$$

where $\beta(u,v)$ is the number of characteristic values of the problem

$$\begin{cases} -\Delta^2 u(t-1) = \mu p\left(v_0^+(t)\right)^{p-1} v, & t \in \mathbb{T}, \\ -\Delta^2 v(t-1) = \mu q\left(u_0^+(t)\right)^{q-1} u, & t \in \mathbb{T}, \\ \Delta u(0) = \Delta u(T) = 0, \\ \Delta v(0) = \Delta v(T) = 0 \end{cases}$$

between 0 and 1. We can calculate $\beta(u,v)$ by the homotopy given in (12). Using Lemma 2, we conclude that

$$\beta(u,v) = 0.$$

Therefore, for sufficiently large R,

$$d\left(\mathrm{Id} - T_{(f_1,g_1)}, (0,R) \times (0,R), 0\right) = \sum_{i=1}^{n} (-1)^{\beta_i(u,v)} = \sum_{i=1}^{n} (-1)^0 \neq 0.$$

Now, consider the following homotopy

$$\begin{cases} -\Delta^2 u(t-1) = \left(v^+(t)\right)^p + (1-\tau) f + \tau f_1, & t \in \mathbb{T}, \\ -\Delta^2 v(t-1) = \left(u^+(t)\right)^q + (1-\tau) g + \tau g_1, & t \in \mathbb{T}, \\ \Delta u(0) = \Delta u(T) = 0, \\ \Delta v(0) = \Delta v(T) = 0, \end{cases} \tag{14}$$

with $0 \leqslant \tau \leqslant 1$. Using the a priori estimate from Lemma 3, we conclude that all solutions of the system (14) are uniformly bounded in $X \times X$. Therefore, for sufficiently large $R > 0$,

$$d\left(\mathrm{Id} - T_{(f,g)}, (0,R) \times (0,R), 0\right) = d\left(\mathrm{Id} - T_{(f_1,g_1)}, (0,R) \times (0,R), 0\right) \neq 0,$$

this completes the proof of Theorem 3. □

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在第一特征值处共振的超线性Neumann问题的可解性

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摘要： 研究一类超线性二阶离散Neumann边值问题

$$\begin{cases}-\Delta^2 u(t-1)=\left(u^+(t)\right)^p+f(t), & t\in[1,T]_{\mathbb{Z}},\\ \Delta u(0)=\Delta u(T)=0,\end{cases} \tag{P}$$

解的存在性，其中 $T\geqslant 2$ 为整数，$p>1$，函数 f 满足 $\sum_{t=1}^{T} f(t)<0$. 利用拓扑度理论，证明了问题(P)至少存在一个解 . 此外，将相关方法推广至具有类似非线性结构的二阶离散方程组 .

关键词： 离散方程;共振;超线性;拓扑度

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